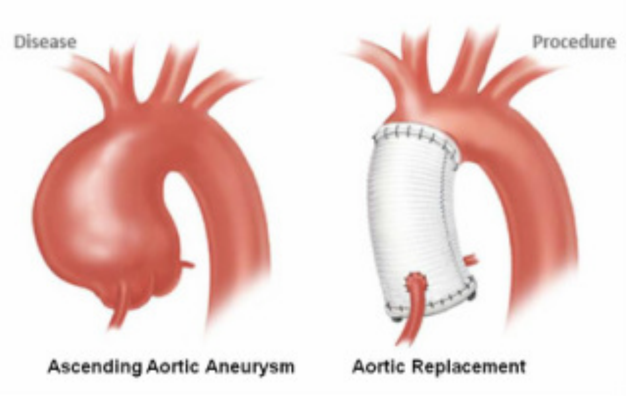
Georg Fantner

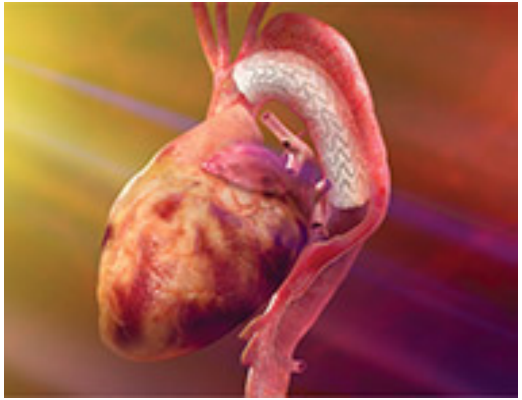
Thin-walled pressure vessel

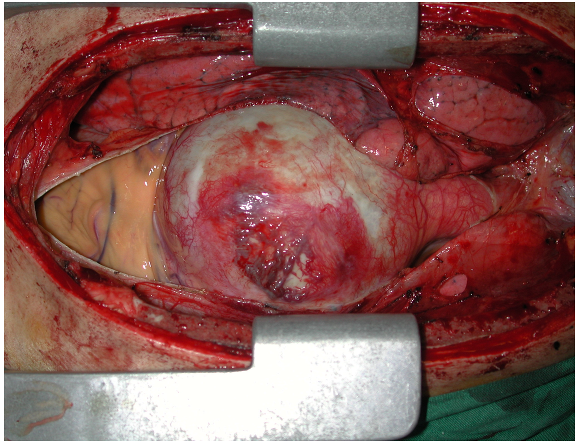
Why a hotdog always ruptures along its length…

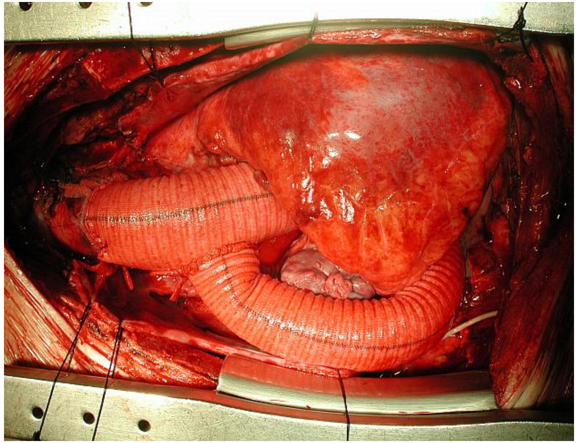
Thin walled pressure vessels

Aortic Aneurysm

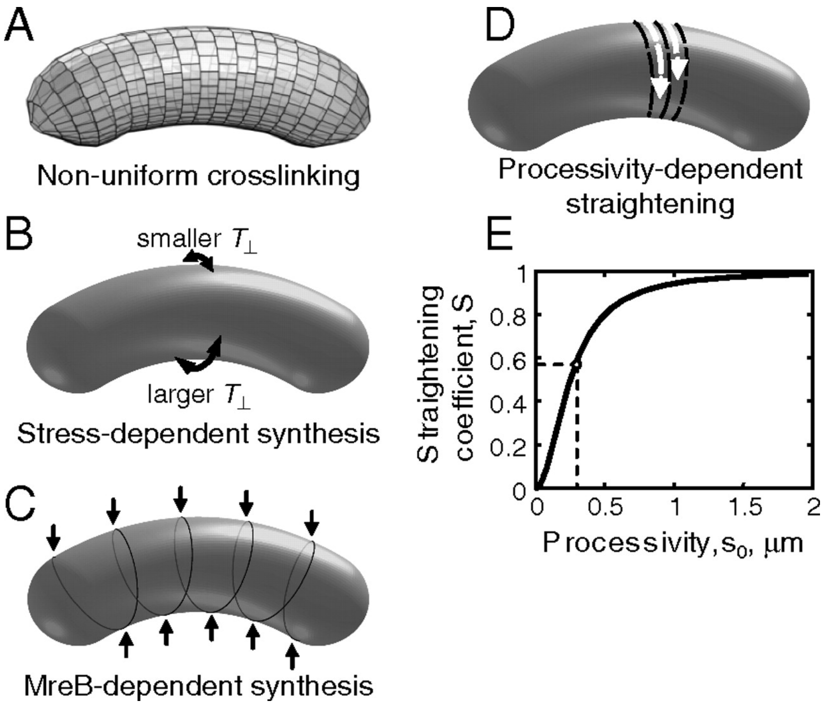


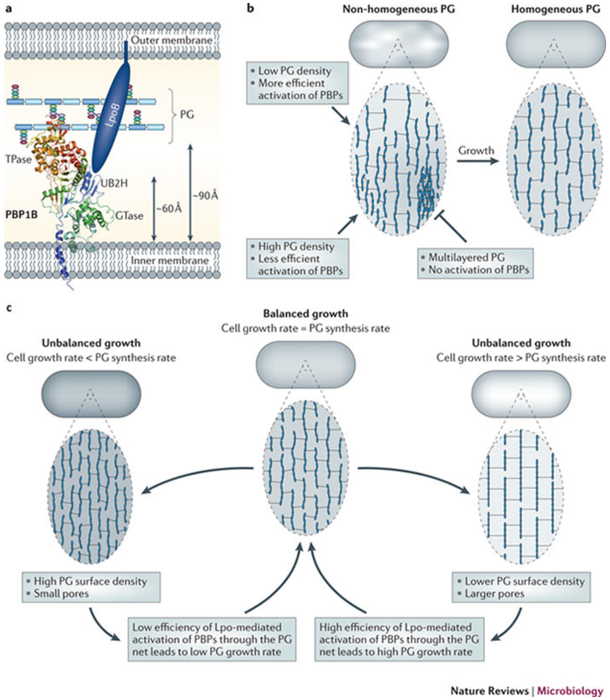






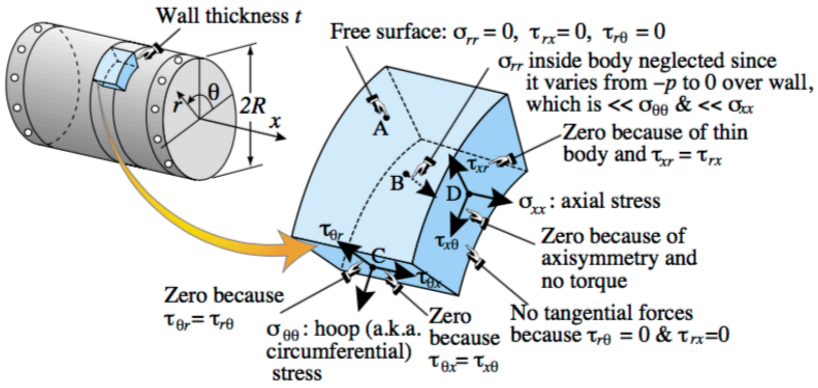
Bacteria as pressure vessels: The effect of Turgor pressure





Thin-walled pressure vessels

- Pressure vessels are generally combinations of spheres, cylinders or ellipsoids, with the task of containing gasses or liquids under pressure.
- We are interested in the stresses that occur in the walls of the pressure vessel.
- We call a pressure vessel *thin walled* if the thickness $t<0.1r_i$ inner radius (examples: boiler, scuba tank, inflated balloon). In this case the wall acts like a membrane and experiences no bending, no significant variation in the stress from the inner to the outer surface.
- We call a pressure vessel thick walled if $t>0.1r_i$ (examples: gun barrel, explosion chamber, high pressure hydraulic presses)



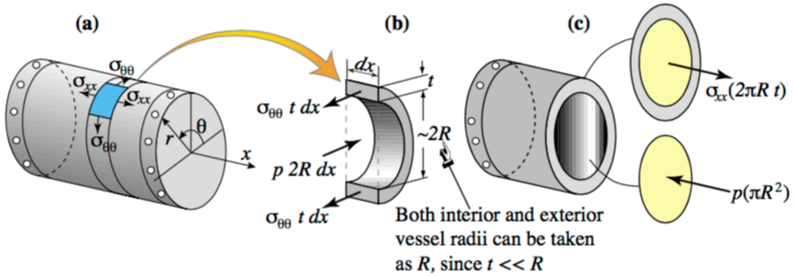
$$\begin{bmatrix} \sigma_{xx} & \tau_{x\theta} & \tau_{xr} \\ \tau_{\theta x} & \sigma_{\theta\theta} & \tau_{\theta r} \\ \tau_{rx} & \tau_{r\theta} & \sigma_{rr} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & 0 & 0 \\ 0 & \sigma_{\theta\theta} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma_{xx} = \text{Axial stress}$$

$$\sigma_{\theta\theta} = \text{Hoop stress}$$

Thin-walled pressure vessels

- We recall: pressure exerts a force/unit area, normal to the area (=parallel to the normal vector of the area)
- This pressure induces a tensile stress in the thin wall.
- The thin wall is in *plane stress!*
- Since the pressure does not apply any shear loads, the shear components on all sides have to be zero.
- In cylindrical coordinates

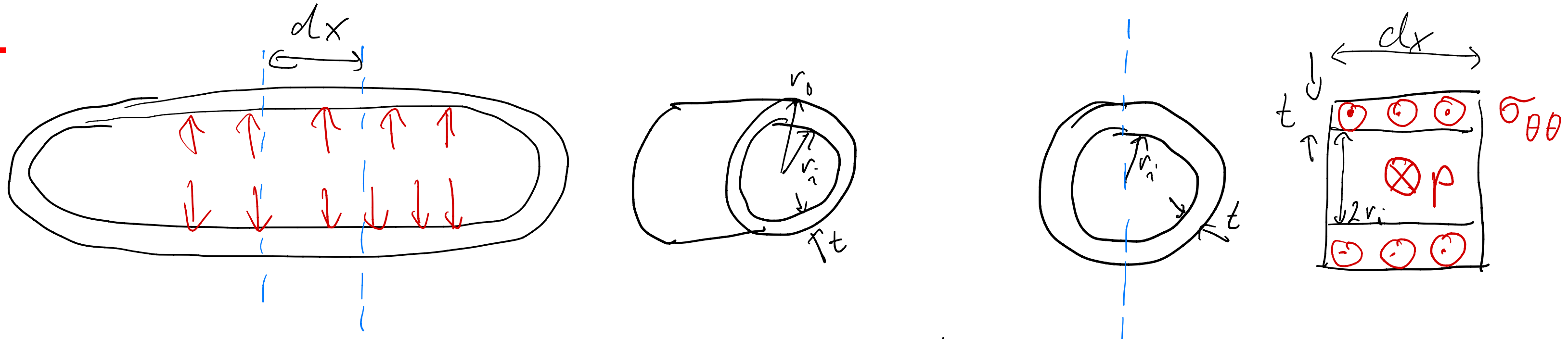


Thin walled pressure vessels

Look at the cross section from the side:

Projected “inner” area D_iL

Wall area: 2 t L



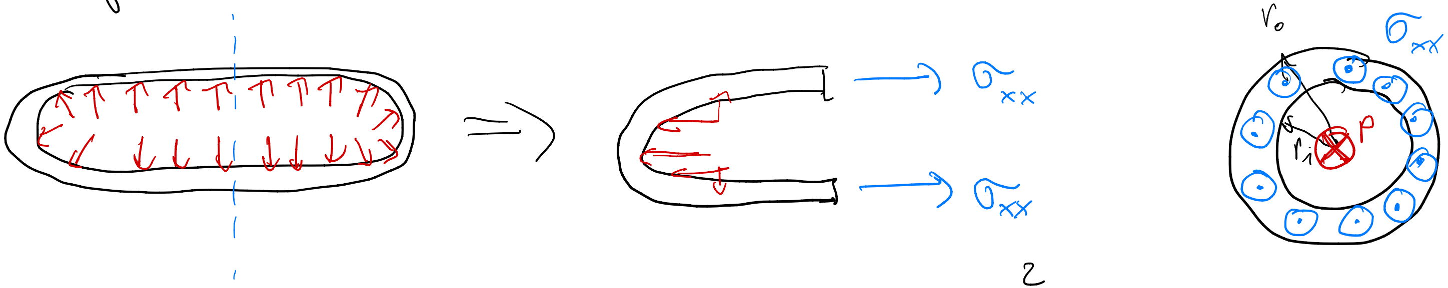
Force due to p: $F_p = p \cdot A = p \cdot 2 r_i \, dx$

Force due to $\sigma_{\theta\theta}$: $F_\sigma = \sigma_{\theta\theta} \cdot 2 t \, dx$

$$p \, \cancel{2} r_i \, \cancel{dx} = \sigma_{\theta\theta} \cdot \cancel{2} t \, \cancel{dx}$$

$$\boxed{\sigma_{\theta\theta} = \frac{p \cdot r_i}{t}} := \sigma_1 \ \ldots \ \text{Hoop stress}$$

Longitudinal direction:



Force due to pressure: $F_p = pA = p\pi r_i^2$

Force due to σ_{xx} : $F_\sigma = \sigma_{xx} A = \sigma_{xx} \cdot \pi (r_o^2 - r_i^2)$

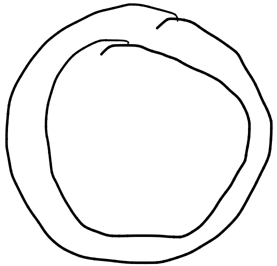
$$p\pi r_i^2 = \sigma_{xx}\pi(r_o^2 - r_i^2)$$

$$\sigma_{xx} = \frac{p r_i^2}{r_o^2 - r_i^2} = \frac{p r_i^2}{(r_o + r_i)(r_o - r_i)}$$

$(r_o + r_i) \approx 2r_i$, $(r_o - r_i) = t$

$$\boxed{\sigma_{xx} = \frac{p r_i}{2t}} := \sigma_2 \ldots \text{ Axial or Longitudinal Stress}$$

SPHERICAL PRESSURE VESSEL



$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

Hoop-stresses

- The tensile stresses around the circumference are called *hoop stresses*. In cylindrical coordinates they would be called:

$$\sigma_{\theta\theta} = \sigma_\theta := \sigma_1$$

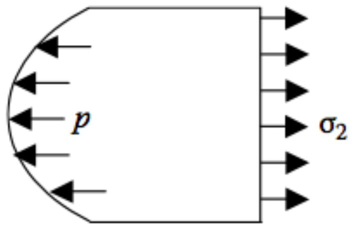
- From the equilibrium of forces in the “out of plane” direction we get:

$$p \cdot r_i \cdot dx = \sigma_{\theta\theta} \cdot 2t \cdot dx$$

- Hoop stress:

$$\boxed{\sigma_1 = \frac{p\,D_i}{2t} = \frac{p\,r_i}{t}}$$

Longitudinal stresses



- *Axial or longitudinal stresses* σ_2: stresses acting along the axis of the pressure vessel.
- The projected cross-section of the end caps is a circle with area:

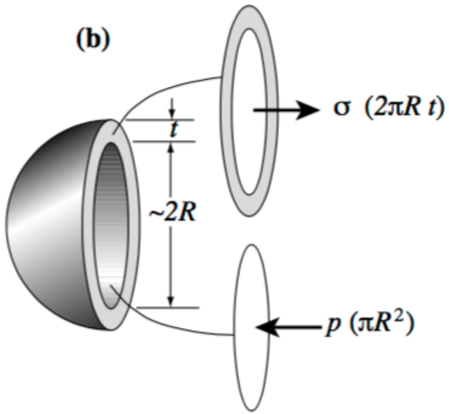
$$A = \pi r_i^2$$

- The force in the longitudinal direction is therefore:
- F_p is balanced by the longitudinal stress $\sigma_{xx}=\sigma_x=\sigma_2$ on the area of the cross-section

$$F_p = p \cdot A = p \cdot \pi\, r_i^2$$
$$F_\sigma = \sigma_x \cdot \pi(r_0^2 - r_i^2)$$

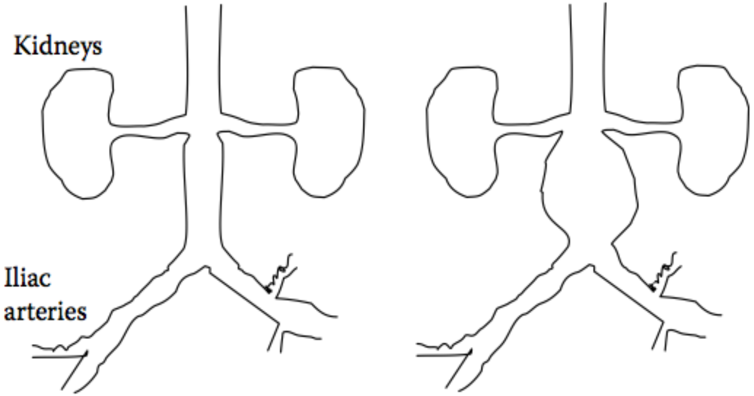
- From force equilibrium:

$$\boxed{\sigma_2 = \frac{pr}{2t}}$$



- For a thin walled <u>spherical</u> pressure vessel:

$$\sigma_1 = \sigma_2 = \frac{pr}{2t}$$

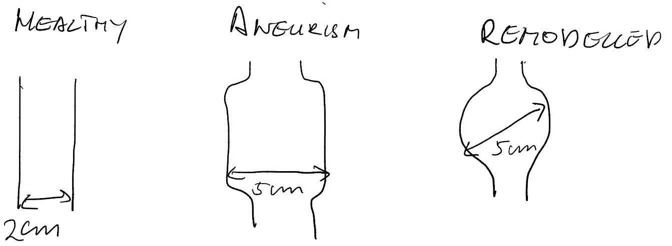


Example: arterial aneurism

Aneurism is a condition where there is ballooning or dilation in a blood vessel.

- Calculate the hoop stress in a healthy artery modeled as a cylinder (r_i=1cm)
- Calculate the hoop stress in a dilated artery modeled as a cylinder (r_i=2.5cm).
- Calculate the hoop stress in a ballooning artery with d=5cm

Assume: thin-walled pressure vessel, blood pressure varies from low(diabolic) to high (systolic) pressure during one heartbeat: $p_{systolic}$=1.6 N/cm^2. The artery has a wall thickness of 1mm.



GIVEN : GEOMETRY : $r = 1\,cm$ for HEALTHY

$r = 2.5\,cm$ FOR ANEURISM

$r = 2.5\,cm$ in REMODELLED CASE

$t = 0.1\,cm$

LOAD $= 1.6\ N/cm^2$

ASKED: σ_1 in ALL CASES.

GOV. PRINC.: THIN WALL PRESSURE VESSEL: $\sigma_1 = \frac{pr}{t}$ $\quad \sigma_2 = \frac{pr}{2t}$

ANSWER:

1) HEALTHY: $\sigma_1 = \frac{pr}{t} = \frac{1.6\ N/cm^2 \cdot 1cm}{0.1\ cm} = 16\ N/cm^2$

2) ANEURISM: $\sigma_1 = \frac{pr}{t} = \frac{1.6\ N/cm^2 \cdot 2.5cm}{0.1\ cm} = 40\ N/cm^2$

3) REMODELLED: $\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{1.6\ N/cm^2 \cdot 2.5cm}{0.2\ cm} = 20\ N/cm^2$

Georg Fantner

Week 8: Transformation of stresses and strains

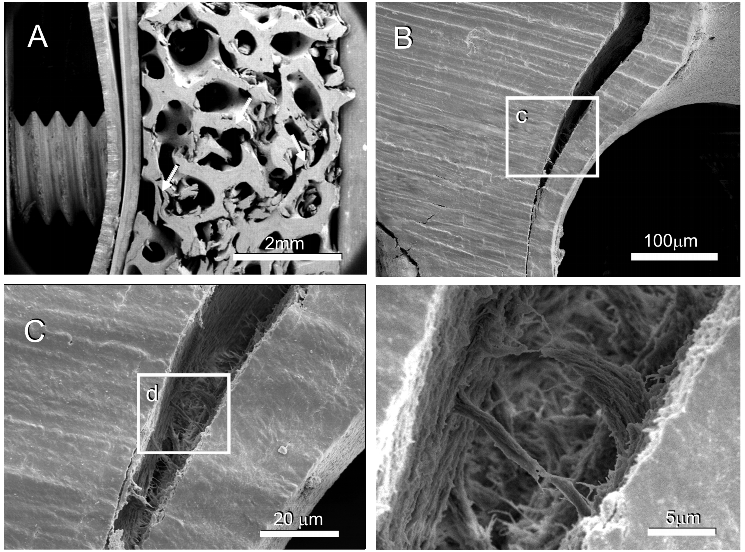
1. Transformation of plane stress
2. Principal and maximum stresses

Transformation of stress and strain

- So far, we've looked at isolated effects of normal and shear stress due to loading by axial forces and shear forces
- We've always looked at what the effects of stresses are along the direction that they are acting on (or in the direction 90degr. from that) to see if we are within the safe stress limits of the material
- However, a combination of normal stress and shear stress can result in much larger normal stresses in a different direction.
- To calculate these maximum stress values and the angles in which they occur, we need to find a way to calculate the stress in any direction that is oriented at an arbitrary angle to our reference axes

Why do we care about stress transformations?

Many materials are not uniformly strong in all directions!

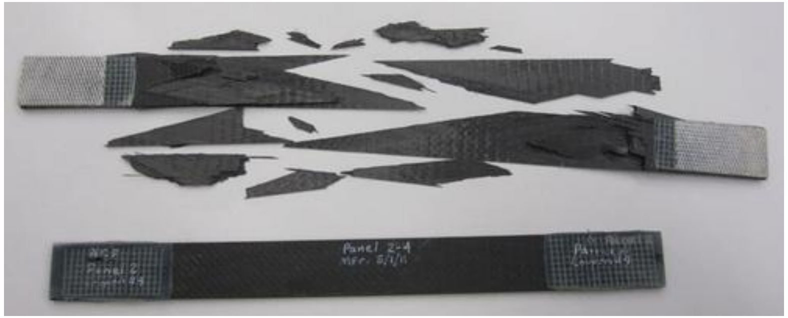


Fantner et al. ***Bone*** *2004*

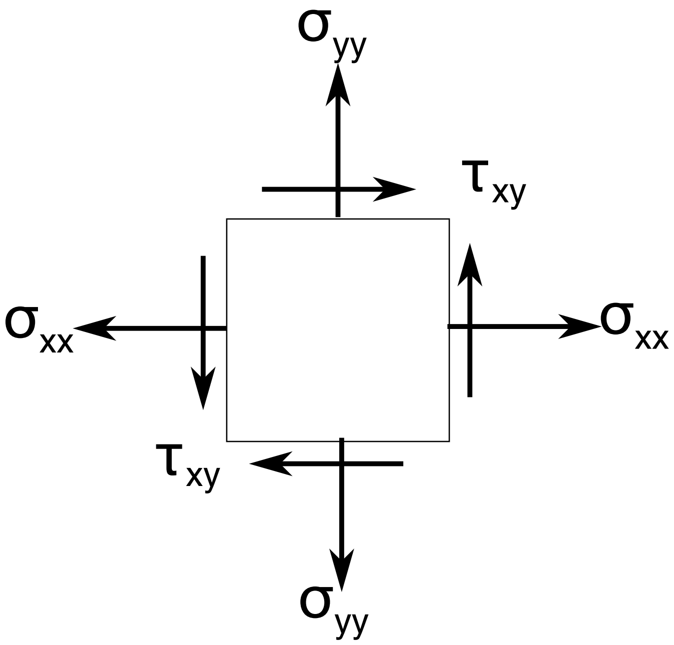
- Bone consists of mineralized collagen fibrils.
- Bone fracture occurs by delamination along the fibrils

Very important for composite materials

Carbon composite is very strong in some direction, much weaker in other directions.



Transformation of stress and strain



What if we have multiple loads acting at the same time?

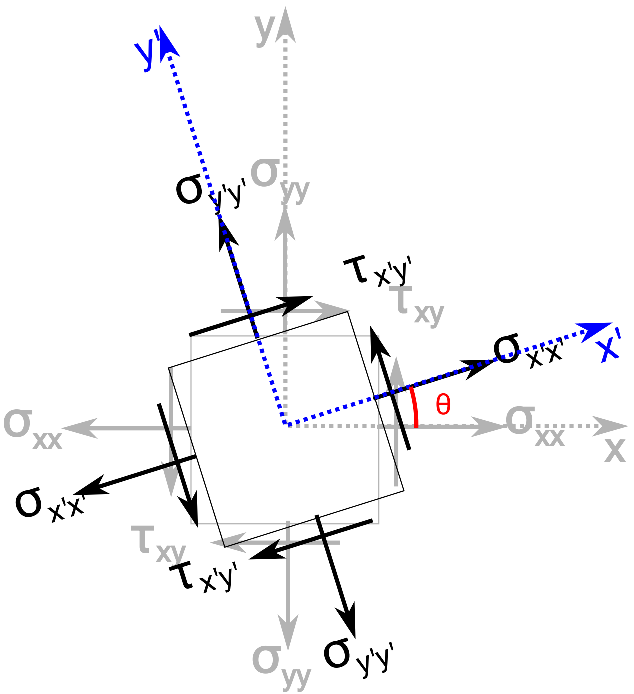
A square sheet of material is loaded in the X and Y direction

Let us assume:

- σ_{xx} is 75% of σ_{max}
- σ_{yy} is 75% of σ_{max}
- τ_{xy} is 75% of τ_{max}

Each of the loads individually do not exceed the fracture limit

But will the plate withstand the combination?

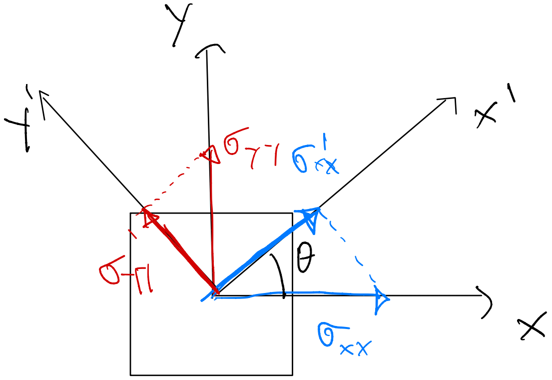


Transformation of stress and strain

We need to find a way to evaluate the stresses and strains in any arbitrary direction.

We already know that we can represent stresses and strains as tensors.

Strictly speaking what we will do is perform a coordinate transform of the tensor to a new coordinate system with the unit vectors in the directions that we are interested in.



FOR A 2D COORDINATE SYSTEM WE CAN USE

$$\overleftrightarrow{Q} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\vec{V}' = \overleftrightarrow{Q}\,\vec{V}$$

$$\overleftrightarrow{T}' = \overleftrightarrow{Q}\,\overleftrightarrow{T}\,\overleftrightarrow{Q}^T$$

We will do matrix coordinate transforms instead

- We can rotate the coordinate system of a vector (x,y) by an angle θ by multiplying with the transformation matrix **Q:**

$$\vec{v}\,' = \mathbf{Q} \cdot \vec{v}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- We can rotate the coordinate system of a tensor by an angle θ by multiplying with the transformation matrix **Q:**

$$\overleftrightarrow{t}\,' = \mathbf{Q} \cdot \overleftrightarrow{t}\,\mathbf{Q}^{\mathbf{T}}$$

$$\begin{pmatrix} x'_{11} & x'_{12} \\ x'_{21} & x'_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

PLANE STRESS: $\sigma_z = 0 \quad \tau_{xz}, \tau_{yz} = 0$

$$\overleftrightarrow{\sigma}' = Q \cdot \overleftrightarrow{\sigma} \cdot Q^T$$

$$\overleftrightarrow{\sigma}' = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{x'x'} & \tau_{x'y'} \\ \tau_{x'y'} & \sigma_{y'y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_{xx}\cos\theta + \tau_{xy}\sin\theta & -\sigma_{xx}\sin\theta + \tau_{xy}\cos\theta \\ \tau_{xy}\cos\theta + \sigma_{yy}\sin\theta & -\tau_{xy}\sin\theta + \sigma_{yy}\cos\theta \end{pmatrix}$$

$$\underline{\sigma_{x'x'}} = \sigma_{xx}\cos^2\theta + \tau_{xy}\sin\theta\cos\theta + \tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$

$$= \underline{\sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta}$$

$$\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \tau_{xy}\sin\theta\cos\theta + \tau_{xy}\sin\theta\cos\theta + \sigma_{yy}\sin^2\theta$$

$$= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

$$2\sin\theta\cos\theta = \sin 2\theta$$
$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$
$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$
$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sigma_{xx}' = \frac{\sigma_{xx}}{2}(1+\cos 2\theta) + \frac{\sigma_{yy}}{2}(1-\cos\theta) + \tau_{xy}\sin 2\theta$$

$$\sigma_{xx}' = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\begin{pmatrix} \sigma_{x'x'} & \sigma_{x'y'} \\ \tau_{x'y'} & \sigma_{y'y'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sigma_{xx}\cos\theta + \tau_{xy}\sin\theta & -\sigma_{xx}\sin\theta + \tau_{xy}\cos\theta \\ \tau_{xy}\cos\theta + \sigma_{yy}\sin\theta & -\tau_{xy}\sin\theta + \sigma_{yy}\cos\theta \end{pmatrix}$$

$$\tau_{xy}' = -\sigma_{xx}\sin\theta\cos\theta + \tau_{xy}\cos^2\theta - \tau_{xy}\sin^2\theta + \sigma_{yy}\sin\theta\cos\theta$$

$$= (\sigma_{yy} - \sigma_{xx})\underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\sin 2\theta} + \tau_{xy}\underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta}$$

$$\boxed{\tau_{xy}' = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta}$$

$$\boxed{\sigma_{yy}' = \frac{\sigma_{xx} + \sigma_{yy}}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta}$$

WE CAN SHOW:

$$\sigma_{xx}' + \sigma_{yy}' = \sigma_{xx} + \sigma_{yy}$$

$\Rightarrow$ STRESS - INVARIANTS

Transformation of plane stress

- The result of all this are the formulas to transform the normal stress and shear stress from the coordinates x,y to x' and y'

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

- We can also show from these formulas that

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$